Equivariant geometry of Alexandrov 3-spaces

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Plan:

1. Alexandrov spaces.
2. Group actions on them ↪.
3. Circle actions on Alexandrov 3-spaces.
4. Applications.
Alexandrov spaces

The model space $M^2_k$ with constant curvature $k \in \mathbb{R}$ is:

(a) $(\mathbb{R}^2, g_0)$, if $k = 0$.
(b) $(\mathbb{S}^2(1/\sqrt{k}), g_0)$, if $k > 0$.
(c) $(\mathbb{H}^2(1/\sqrt{-k}), g_0)$ if $k < 0$.

$$\text{diam}(M^2_k) = \begin{cases} \pi/\sqrt{k} & \text{si } k > 0; \\ \infty & \text{si } k \leq 0. \end{cases}$$
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Alexandrov spaces $(X, d)$ are **length spaces**, i.e. metric spaces such that

$$\text{dist}(p, q) = \inf_{\gamma \in \Omega_{pq}} \text{Length}(\gamma)$$

$$\text{Length}(\gamma) = \sup \sum_{i=1}^{n} \text{dist}(\gamma(t_i), \gamma(t_{i+1}))$$
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They are defined by comparing geodesic triangles to those of the model space:

\[\triangle pqr: \text{ geodesic triangle in } X\]

\[k \in \mathbb{R}\]

\[\overline{\triangle pqr}: \text{ comparison triangle } = \text{ triangle in } M^2_k \text{ whose sides have the same lengths as the corresponding sides of } \triangle pqr.\]
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$\triangle \overline{pqr}$: comparison triangle = triangle in $M^2_k$ whose sides have the same lengths as the corresponding sides of $\triangle pqr$.

$\triangle \overline{pqr}$ exists and is unique if $k \leq 0$ or $k > 0$ and

$$d(p, q) + d(p, r) + d(q, r) < \frac{2\pi}{\sqrt{k}}.$$
Definition (T-property) Given $\triangle pqr$ in $X$ and a comparison triangle $\triangle pqr$, for any $s \in [qr]$ and the corresponding point $\bar{s} \in [\bar{q}\bar{r}]$

\[ \text{dist} (p, s) \geq \text{dist} (\bar{p}, \bar{s}). \]
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A length space has curv $X \geq k$ if each $x \in X$ has a neighborhood $U$ in which the T-property holds for every $\triangle pqr \subset U$. 
Alexandrov spaces (cont’d)

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A length space has $\text{curv} X \geq k$ if each $x \in X$ has a neighborhood $U$ in which the T-property holds for every $\triangle pqr \subset U$.

**Definition** An **Alexandrov space** is a complete, locally compact length space with $\text{curv} \geq k$ for some $k \in \mathbb{R}$.
Examples & constructions

(i) Riemannian manifolds with $\sec \geq k$. 

(ii) Euclidean cones 
$$K(Y) := Y \times [0, \infty) / Y \times \{0\}$$ 
$$d((p, t), (q, s)) := \sqrt{t^2 + s^2 + 2ts \cos d_Y(p, q)}$$ 
$$\text{curv}_Y \geq 1 \Rightarrow \text{curv}_K(Y) \geq 0.$$

(iii) Suspensions 
$$\text{Susp}(Y) = Y \times [0, \pi] / Y \times \{0\}, Y \times \{\pi\},$$ 
$$\cos d((p, t), (q, s)) := \cos t \cos s + \sin t \sin s \cos d_Y(p, q).$$ 
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(iv) Quotients of isometric actions. 
$$G \text{ compact Lie group} \curvearrowright \text{by isometries} \text{on a Riemannian manifold } M \text{ with } \sec \geq k.$$ 
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(vi) Gluings:
If \( X, Y \) are Alexandrov spaces with \( \text{curv} \geq k \) and we have an isometry \( \partial X \to \partial Y \) then \( X \cup_{\partial} Y \) is an Alexandrov space of \( \text{curv} \geq k \).

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\( G \) compact Lie group \( \curvearrowright \) by isometries on a Riemannian manifold \( M \) with \( \text{sec} \geq k \). Then \( \text{curv} M/G \geq k \).
Properties

(i) Geodesics do not branch.

\[
\begin{array}{c}
\text{Diagram of geodesics not branching.}
\end{array}
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(iv) Angles between geodesics at the same point are well defined.
\[ \alpha(t_1) \quad \alpha(t) \]

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$$\angle(\alpha, \beta) = \lim_{t_i, s_i \to 0} \{\angle \alpha(t_i)p\beta(s_i)\}.$$
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Tangent direction at $p$: geodesics/angle zero

$(S_p, \angle)$ possibly noncomplete metric space.

$(\Sigma_p, \angle)$ metric completion is called the space of directions of $X$ at $p$. 

Not an Alexandrov space:
Properties (cont’d)

(v) Thm. (Burago, Gromov, Perelman)
1. $\Sigma_p$ is an Alexandrov space with $\text{curv} \geq 1$
2. $\dim \Sigma_p = \dim X - 1$
3. $\Sigma_p$ is homeomorphic to $\mathbb{S}^{n-1}$ on a dense set. (topologically regular points)
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(vi) Thm. (Conical neighborhood (Perelman))
Every $p \in X$ has a neighborhood pointed-homeomorphic to the cone over $\Sigma_p$. 

(vii) Boundary of an Alexandrov space:
1. If $\dim X = 1$, $\partial X$ is the topological boundary.
2. For $\dim X = n > 1$, $p \in \partial X$ if and only if $\partial \Sigma_p \neq \emptyset$.
$\partial X$ is a closed subset of codim. 1.
$\partial X \subseteq \mathbb{S}(X)$. If $\partial X = \emptyset$, then $\text{codim} \mathbb{S}(X) \geq 3$. And so:
· $\dim X = 1, 2 \Rightarrow X$ is homeomorphic to a topological manifold.
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$\partial X \subset S(X)$. If $\partial X = \emptyset$, then $\text{codim} S(X) \geq 3$. And so:
- $\dim X = 1, 2 \implies X$ is homeomorphic to a topological manifold.
If $X$ is closed and $\dim X = 3$, then $\dim \Sigma_p = 2$.

**Bonnet-Myers Thm** $\implies \pi_1 \Sigma_p$ is finite
$\implies \Sigma_p \cong S^2$ or $\Sigma_p \cong \mathbb{R}P^2$. 
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**Conical Nhgb Thm & codim. of $S(X)$**
$\implies$ Finite number of points with $\Sigma_p \cong \mathbb{R}P^2$.
$\implies X \cong M^3 \cup_{i=1}^{2s} K(\mathbb{R}P^2)$. 

Other description: Grove-Wilking, Harvey-Searle: non-manifold $X^3$, then there exist:
- Alexandrov manifold $\tilde{X}^3$ (orientable branched double cover)
- Orientation-reversing isometric involution $\iota: \tilde{X} \to \tilde{X}$ with only fixed points such that $X$ is isometric to $\tilde{X}/\iota$.

Ramification locus $\to$ topologically singular points.
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such that $X$ is isometric to $\tilde{X}/\nu$.
Ramification locus $\to$ topologically singular points
Thm. (Fukaya, Yamaguchi)
\(\text{Isom}(X)\) is a Lie group.
\(X\) compact \(\implies \text{Iso}(X)\) is compact.

Measures of the size of \(\text{Iso}(X)\):
- Symmetry degree: \(\dim \text{Iso}(X)\)
- Symmetry rank: \(\text{rankIso}(X)\)
- Cohomogeneity: \(\dim X/G\) where \(G \leq \text{Iso}(X)\).

Cohomogeneity 0: Thm. (Berestovskii)
A homogeneous Alexandrov space is isometric to a Riemannian manifold.

Cohomogeneity 1: Here, \(X/G\) must be either a circle or an interval.
Thm. (Galaz-García, Searle)
- If $X/G$ is a circle then $M$ is equivariantly homeomorphic to a fiber bundle over $\mathbb{S}^1$ with fiber $G/H$ and structure group $N(H)/H$. In particular $X$ is a manifold.
- If $X/G \cong [-1, 1]$, then there is a group diagram

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Group actions on Alexandrov spaces (cont’d)

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\[
\begin{array}{ccc}
G & & \\
\downarrow j_- & & \uparrow j_+ \\
L_- & & L_+ \\
\downarrow i_- & & \uparrow i_+ \\
H & & \\
\end{array}
\]

- $L_\pm$ istropies at $\pm 1$, $H$ principal isotropy, $L_\pm/H$ are isometric to a homogeneous space with $\sec > 0$.
- $X$ is the union of two fiber bundles with base $G/L_\pm$ and fiber $K(L_\pm/H)$.

Note: This allows them to classify topologically cohomogeneity one Alexandrov spaces up to dimension 4. The only non-manifold one in dim. 3 is $\text{Susp}(\mathbb{R}P^2)$. 
We focus on the case $X$ closed, $\dim X = 3$ and effective action, i.e. principal isotropy is $e$. Because of the dimensions $G = S^1$. 

(i) The action leaves the topologically regular and topologically singular sets invariant.

(ii) The possible isotropies are the closed subgroups of $S^1$: $e$, $\mathbb{Z}_k$ and $S^1$ itself.

(iii) The manifold case was done by Orlik, Raymond: $M/G$ is a 2-manifold carrying a set of topological/equivariant invariants.
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Cohomogeneity 2 Alexandrov spaces

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Circle actions on 3-manifolds

Analysis of nghbs of the orbits and associate invariants:

\[ b \in \mathbb{H}_2(O^*, \mathbb{Z}) \]

\( O \) is the principal stratum. Open and dense by the Principal orbit Thm. (Alexandrov version by: Galaz-García, Guijarro).

\( g \) is the genus of \( M^* \).

\( \varepsilon \in \{ o, n \} \) depending of orientabilities.

\( f \) is the number of boundary cpts. of fixed points.

\( t \) is the number of boundary cpts. of isotropy \( \mathbb{Z}_2(\alpha_i, \beta_i) \) are the Seifert invariants of orbits with isotropy \( \mathbb{Z}_k \).
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- \((\alpha_i, \beta_i)\) are the Seifert invariants of orbits with isotropy \( \mathbb{Z}_k \).
Cohomogeneity 2 Alexandrov spaces

Thm. (Orlik, Raymond)

- A circle action on a closed $M^3$ is uniquely determined up to equivariant homeomorphism by the set of invariants
  $$\{ b; (\varepsilon, g, f, t); \{(\alpha_i, \beta_i)\}_{i=1}^n \}$$

- If $f > 0$ then $M^3$ is equivariantly homeomorphic to
  $$M_{(\varepsilon, g, f, t)} \# L(\alpha_1, \beta_1) \# \ldots \# L(\alpha_m, \beta_m),$$
  where $M_{(\varepsilon, g, f, t)}$ is
  $$N \# (\#_{i=1}^l \mathbb{S}^2 \times \mathbb{S}^1) \# (\#_{i=1}^t \mathbb{R}P^2 \times \mathbb{S}^1)$$
  and
  $$N \cong \mathbb{S}^3 \quad \text{or} \quad \mathbb{S}^2 \tilde{\times} \mathbb{S}^1.$$
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Thm. (–, 2014)
- A circle action on a closed Alexandrov space $X^3$ is uniquely determined by the set of invariants
  $$\{b; (\varepsilon, g, f, t); \{ (\alpha_i, \beta_i) \}; (r_1, \ldots, r_s) \}$$
  $$r_i$$ even non-negative integers.
- $X$ is equivariantly homeomorphic to
  $$M \# \left(\#_{i=1}^s \text{Susp}(\mathbb{R}P^2)\right)$$
  where $M = \{b; (\varepsilon, g, f + s, t); \{ (\alpha_i, \beta_i) \}_{i=1}^n \}$.
Sketch of proof.

The analysis of the action at topologically regular points is that of Orlik-Raymond. We need to see what happens at topologically singular points.

(i) Top. singular points are fixed points.
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**Thm. Slice Theorem (Harvey-Searle)**

Let a compact Lie group $G$ act by isometries on an Alexandrov space $X$, then a small nghbd. of the orbit $G/G_p$ is equivariantly homeomorphic to $G \times_{G_p} K(S_p^\perp)$. ($S_p^\perp$ is the subset of $\Sigma_p$ orthogonal to the orbit).
Sketch of proof.

The analysis of the action at topologically regular points is that of Orlik-Raymond. We need to see what happens at topologically singular points.

(i) top. singular points are fixed points.

(ii) Slice Thm $\implies$ action commutes with cone construction at $p$ and $\implies K(\mathbb{R}P^2)/S^1 \cong K(\mathbb{R}P^2/S^1)$. 

(iii) Only one $S^1$ action on $\mathbb{R}P^2$; the one induced by rotations on $S^2$. 

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Sketch of proof (cont’d)

Orbit space in the Alexandrov case:

- **New invariants**: \( (r_1, \ldots, r_s) \): Number of top. sing. points on each boundary component.
(iv) For each action we can “read off” the invariants.

(v) Now, given a set of invariants we want an Alexandrov space with an $S^1$-action with those invariants and prove that two Alexandrov spaces with the same invariants are equivariantly homeomorphic.

(vi) First we think about the case $X/G$ is homeomorphic to a 2-disk and no isolated $\mathbb{Z}_k$ orbits.
(vii) We can construct a cross-section $X/S^1 \to X$ to $\pi : X \to X/S^1$:

- $P \to P^*$ is a trivial principal $S^1$-bundle.
- Extend the cross-section to $U^*_{RF}$ and $U^*_{SE}$ “radially”.
- Extend to the conical neighborhoods by “copying the base”.

$\pi : X \to X/S^1$
(viii) With this, if there are two Alexandrov spaces $X$, $Y$ whose orbit space is a 2-disk and with same invariants we can give an equivariant homeomorphism via the sections:

$$X - \cdots - \varphi \rightarrow Y$$

$$\pi_1 \rightarrow X/S^1 \cong Y/S^1 \rightarrow \pi_2$$

It follows by induction that this is true for any number of top. singular points.
(ix) This shows that \(\text{Susp}(\mathbb{R}P^2)\# \ldots \# \text{Susp}(\mathbb{R}P^2)\) is the only Alexandrov 3-space with such an orbit space.
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\[ M = \{ b; (\varepsilon, g, f + s, t); \{(\alpha_i, \beta_i)\}_{i=1}^n \} \]\nhas at least $s$ circles of fixed points so we can take equivariant connected sums with $\text{Susp}(\mathbb{R}P^2) \# \ldots \# \text{Susp}(\mathbb{R}P^2)$. 

\[ \text{(x) The manifold} \]

\[ \text{Sketch of proof (cont’d)} \]
The decomposition in equivariant connected sums makes it plausible to compute topological invariants and obtain the following application:

- Recall that $X$ is aspherical if $\pi_q(X) = 0$, $q > 1$. 

**Borel conjecture:** If two closed aspherical manifolds are homotopy equivalent then they are homeomorphic. (False in the smooth category because of exotic manifolds, but true for $n \leq 3$ at least).
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Mayer-Vietoris $\Rightarrow$

\[ H_2(M \# (\#_{i=1}^s \text{Susp}(\mathbb{R}P^2))) \cong H_2(M) \oplus H_2(\#_{i=1}^s \text{Susp}(\mathbb{R}P^2)) \]
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- Therefore, the only aspherical Alexandrov spaces with an isometric $S^1$ action are 3-manifolds, and the Borel conjecture is true there.
An isometric local circle action is a decomposition of $X$ into disjoint curves (possibly points) such that each curve has a nhgb. with an isometric circle action whose fibers are the curves of the decomposition.

Classified in the manifold case by Fintushel, Orlik-Raymond. In the Alexandrov case we obtained

$$X \text{ is equivariantly homeomorphic to } M \# \text{Susp}(\mathbb{R}P^2)\# \cdots \# \text{Susp}(\mathbb{R}P^2),$$

where $M$ is the closed 3-manifold determined by the set of invariants

$$(b; \varepsilon, g, (f, k_1), (t, k_2); \{(\alpha_i, \beta_i)\}_{i=1}^n)$$

in the manifold case.

Thm (Galaz-García, –, 2016)

- An isometric local circle action on $X$ is determined up to equivariant homeomorphism by

$$(b; \varepsilon, g, (f, k_1), (t, k_2); \{(\alpha_i, \beta_i)\}; (r_1, r_2, \ldots, r_s)).$$
Mitsuishi-Yamaguchi considered collapsing sequences of closed Alexandrov 3-spaces, i.e.

$$X_i \to_{GH} Y$$

with $\text{diam}(X_i) \leq D$, $\text{curv} X_i \geq -1$ and $\text{dim} Y < 3$.

When $\text{dim} Y = 2$ and $\partial Y = \emptyset$, they obtained that $X_i$ is a kind of generalized Seifert fiber space admitting singular interval fibers at some topologically singular points.

Thm (Galaz-García, –, 2016) If the sequence does not have singular interval fibers, the collapse occurs along the fibers of an isometric local circle action on the $X_i$ for $i$ big enough.
An Alexandrov 3-space is **geometric** if it is a quotient of the corresponding Thurston geometry by some cocompact lattice.

We will say a closed Alexandrov 3-space is **irreducible** if every embedded sphere bounds a 3-ball and if the space has topologically singular points we further require that every $\mathbb{R}P^2$ bounds a $K(\mathbb{R}P^2)$. 

**Theorem (Galaz-García, Guijarro –, 2016)**

For any $D > 0$ there exists $\varepsilon(D) > 0$ such that if $X$ is a closed irreducible Alexandrov 3-space with diam $D$ and $\text{vol} \ X < \varepsilon$ then $X$ is geometric.

**Sketch of part of the proof:** Assume that this is not the case. Then for a sequence $\varepsilon_i \to 0$, there is a sequence $X_i$ converging in $\text{GH}$ to $Y$.

**Case dim** $Y = 2$, $\partial Y = \emptyset$. Then by Mitsuishi-Yamaguchi’s work, $X_i$ is a generalized Seifert fiber space. If $X_i$ does not have singular interval fibers, the collapse occurs along the fibers of a local action $= \Rightarrow$ decomposition into equivariant connected sums. $= \Rightarrow$ condition of irreducibility rules out everything but one connected summand, which then is geometric.
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Local circle actions and Thurston geometries (cont’d)

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**Thm. (Galaz-García, −, 2016)** An isometric local circle action on a non-manifold Alexandrov 3-space can be lifted to an isometric local circle action on its branched double cover.

The branched double cover doesn’t have singular fibers. Although $\tilde{X}$ might not be irreducible, it is possible to still see that we can rule out everything but one summand.
Thank you!